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Topological Invariant in Riemann–Cartan Manifold and Space-Time Defects

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In a Riemann–Cartan manifold a topological invariant is constructed in terms of the torsion tensor. Using the ϕ-mapping method and the complete decomposition of the gauge potential, the topological invariant is extricated from a strong restrictive condition and is quantized in units of an elementary length. This topological invariant is linked to the first Chern class and its inner structure is labeled by a set of winding numbers. In the early universe, by extending to a gauge parallel basis in internal space and four analogous topological invariants, the space-time defects are formulated in an invariant form and are quantized naturally in units of the Planck length.

1. INTRODUCTION

In recent years, a great deal of work on spin and torsion has been done by many physicists [1–4]. Though it has been common to include intrinsic spin with gravitation [5–7] and to relate spin to the torsion tensor [8–10], the quantization of the gravitational field and the mechanism of generation of torsion in physics and geometry [11] are not very clear. In recent papers, Ross [12] and De Sabbata [13] investigated these problems from the viewpoint of space-time defects which are described by the integral

$$l^{\lambda} = \oint T^{\lambda}_{\mu\nu} dx^{\mu} \wedge dx^{\nu}, \qquad T^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{[\mu\nu]} \tag{1}$$

where $T^{\lambda}_{\mu\nu}$ is the nonzero torsion tensor in Riemann–Cartan manifold and $\Gamma^{\lambda}_{\mu\nu}$ an asymmetric affine connection. In a discussion of the importance of

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spin and torsion in the early universe De Sabbata [13] assumed that the integral l^λ is quantized in units of the Planck length L_p $(=\sqrt{\hbar G/c^3})$, i.e.,

$$l^\lambda = \oint T^\lambda_{\mu\nu} dx^\mu \wedge dx^\nu = nL_p \tag{2}$$

from which the author defined time at the quantum geometric level through the fourth component as

$$t = \frac{1}{c} \oint T dA = nT_p, \qquad T_p = L_p/c \tag{3}$$

where n is an integer and c the velocity of light. Although the integral (1) and the hypothesis (2) are analogous to the geometrical description of dislocations (defects) in crystals and the well-known Bohr–Sommerfeld relation $\oint pdq = n\hbar$, they violate the general coordinate invariance, and l^λ is not even a vector at all. Under the general coordinate transformation $x \to x'$ with the inverse $x' \to x$, $\Gamma^\lambda_{\mu\nu}$, $T^\lambda_{\mu\nu}$, and l^λ obey

$$\Gamma^{\lambda'}_{\mu'\nu'} = \frac{\partial x^{\lambda'}}{\partial x^\lambda} \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\nu}{\partial x^{\nu'}} \Gamma^\lambda_{\mu\nu} + \frac{\partial x^{\lambda'}}{\partial x^\lambda} \frac{\partial^2 x^\lambda}{\partial x^{\mu'} \partial x^{\nu'}}, \qquad T^{\lambda'}_{\mu'\nu'} = \frac{\partial x^{\lambda'}}{\partial x^\lambda} \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\nu}{\partial x^{\nu'}} T^\lambda_{\mu\nu}$$

$$l^{\lambda'} = \oint T^{\lambda'}_{\mu'\nu'} dx^{\mu'} \wedge dx^{\nu'} = \oint \frac{\partial x^{\lambda'}}{\partial x^\lambda} T^\lambda_{\mu\nu} dx^\mu \wedge dx^\nu \neq \frac{\partial x^{\lambda'}}{\partial x^\lambda} l^\lambda$$

So, l^λ is not observable. In ref. 14 we reconstructed the integral (1) in an invariant form and quantized it at the topological level by means of the vierbein theory and a topological invariant, which had been successfully used in the gauge field theory of dislocation and disclination continuum [15, 16] and the geometrization of Planck's constant [17]. In this paper, we will study the origin of the topological invariant in analogy with the theory of magnetic monopoles and extend the invariant to a more general case in terms of the complete decomposition of $U(1)$ and $SO(2)$ gauge potentials. In units of an elementary length, the topological invariant is quantized rigorously and its inner structure is also studied naturally through a set of topological quantum numbers. In the early universe, the introduction of the Planck length L_p to the space-time defects has a profound significance in general relativity and quantum theory, which may be important in the early universe because of spontaneous symmetry breaking [17].

2. A TOPOLOGICAL INVARIANT IN RIEMANN–CARTAN MANIFOLD

In vierbein theory, the torsion tensor can be expressed by

$$T^A_{\mu\nu} = D_\mu e^A_\nu - D_\nu e^A_\mu \tag{4}$$

where e^A_μ (μ, $A = 1, 2, 3, 4$) is the vierbein field and

$$D_\mu = \partial_\mu - \omega_\mu(x), \qquad \omega_\mu = \frac{1}{2}\omega^{AB}_\mu I_{AB}$$

is the gauge covariant derivative, in which ω^{AB}_μ stands for the spin connection and I_{AB} the generator of the Lorentz group. Since the indices μ and A belong to different spaces, the vierbein e^A_μ obeys two kinds of transformations: one is the general coordinate transformation $x^\mu \to x^{\mu'}$ with the inverse $x^{\mu'} \to x^\mu$, under which

$$e^A_{\mu'} = x^\mu_{\mu'} e^A_\mu, \qquad x^\mu_{\mu'} = \frac{\partial x^\mu}{\partial x^{\mu'}}$$

The other is the local Lorentz transformation $L^A_B(x)$,

$$e'^A_\mu = L^A_B e^B_\mu, \qquad L^C_A(x) L^C_B(x) = \delta_{AB}$$

which is an orthogonal gauge transformation. In order to construct a topological invariant in Riemann–Cartan manifold U_4, as in ref. 15, we can define a gauge parallel vector in internal space, whose existence is closely related to the geodesic $\gamma(s)$,

$$\frac{du^\lambda}{ds} + \Gamma^\lambda_{\mu\nu} u^\mu u^\nu = 0, \qquad u^\mu = \frac{dx^\mu}{ds} \tag{5}$$

which can be further written in the covariant derivative notation [18]

$$\nabla_\mu u^\lambda = \partial_\mu u^\lambda + \Gamma^\lambda_{\mu\nu} u^\nu = 0$$

where ds is the element of length of $\gamma(s)$. Using $\omega^{AB}_\mu = (\nabla_\mu e^A_\nu) e^{\nu B}$, we find that the above covariant notation multiplied by e^B_λ gives

$$D_\mu u^A = 0, \qquad u^A = e^A_\lambda u^\lambda$$

which means $u^A(x)$ is a gauge parallel vector along the geodesic $\gamma(s)$. Though the vector u^λ is defined only at points of $\gamma(s)$, it can be extended to a vector field on a neighborhood of any point of $\gamma(s)$, which leads to $u^A(x)$ also a gauge parallel vector field on this neighborhood [19, 20]. The projection of the torsion tensor (4) along $u^A(x)$ is [15]

$$T_{\mu\nu} = T^A_{\mu\nu} u^A = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{6}$$

where $A_\mu = e^A_\mu u^A$ is the $U(1)$ gauge potential. This shows that $T_{\mu\nu}$ can be expressed in terms of A_μ just like the curvature on $U(1)$ principal bundle with base manifold U_4, i.e., the $U(1)$ gauge field strength, which is invariant for the $U(1)$-like gauge transformation

$$A'_\mu(x) = A_\mu(x) + \partial_\mu \Lambda(x) \tag{7}$$

where $\Lambda(x)$ is an arbitrary function. In analogy with the theory of magnetic monopoles, one has a current j^μ defined by

$$j^\mu := \nabla_\nu T^{*\mu\nu} = \frac{1}{\sqrt{-g}} \partial_\nu(\sqrt{-g}T^{*\mu\nu}) \tag{8}$$

where

$$T^{*\mu\nu} = \frac{1}{2} \frac{\varepsilon^{\mu\nu\lambda\rho}}{\sqrt{-g}} T_{\lambda\rho} \tag{9}$$

is the covariant dual tensor of $T_{\mu\nu}$. It is obvious that the current j^μ is identically conserved,

$$\nabla_\mu j^\mu = \frac{1}{\sqrt{-g}} \partial_\mu(\sqrt{-g}j^\mu) = 0$$

The conserved quantity Q is given by

$$Q = \int_\sigma j^\mu \sqrt{-g} d\sigma_\mu$$

where σ is the spacelike Cauchy surface [21, 22]. Making use of (8), (9), and the Gauss theorem, we can change the quantity Q into

$$Q = \oint_\Sigma \frac{1}{2} T_{\mu\nu} dx^\mu \wedge dx^\nu \tag{10}$$

where Σ is a closed 2-surface with intrinsic coordinates $u = (u^1, u^2)$ and $x^\mu = x^\mu(u^1, u^2)$. The integral in (10) is quite different from that of De Sabbata in (1). Under the general coordinate and local Lorentz transformations, one can prove

$$Q' = \oint_\Sigma \frac{1}{2} T'_{\mu'\nu'} dx^{\mu'} \wedge dx^{\nu'} = \oint_\Sigma \frac{1}{2} T_{\mu\nu} dx^\mu \wedge dx^\nu = Q$$

by using

$$u^{\lambda'} = \frac{\partial x^{\lambda'}}{\partial x^\lambda} u^\lambda, \qquad u'^A = L^A_B u^B, \qquad D'_{\mu'} e'^A_{\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\nu}{\partial x^{\nu'}} L^A_B D_\mu e^B_\nu$$

$$T'^A_{\mu'\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\nu}{\partial x^{\nu'}} L^A_B T^B_{\mu\nu}, \qquad T'_{\mu'\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\nu}{\partial x^{\nu'}} T_{\mu\nu}$$

That is to say, Q has the general coordinate and local Lorentz invariances. So, Q is independent of the choice of coordinate system. In fact, the integral Q in (10) is associated with the first Chern class, which means the quantity Q is a topological invariant [14]. From (6) and (10), the meaning of this topological invariant is the total projection of the torsion tensor $T^A_{\mu\nu}$ on the closed surface Σ. Similar to magnetic monopoles, Q can be called the total torsion charge of the Riemann–Cartan manifold, which is considered to be the source of spin of a physical system [13]. In dislocation continuum theory, it is well known that $T_{\mu\nu}$ represents the dislocation density and Q corresponds to the total projection of the Burgers vector on a surface. In our previous work, this topological invariant has been successfully used to link the topology of dislocations and disclinations to their geometrical description in the gauge field theory of continuum and to construct a new geometrization of Planck constant $\hbar$ at the topological level. One can see easily that Q has the dimension of length.

3. THE DECOMPOSITION OF $U(1)$ AND $SO(2)$ GAUGE POTENTIALS

On Σ a $U(1)$ gauge transformation is equivalent to a two-dimensional rotation and $A_\mu(x)$ corresponds to the $SO(2)$ gauge connection $\omega_\mu^{ab}(x)$ [15, 16]. This relationship can be expressed as follows:

$$\omega_\mu^{ab}(x) = -\frac{2\pi}{L} A_\mu \varepsilon^{ab}, \qquad a, b = 1, 2 \tag{11}$$

where L is a length-dimensional constant that is introduced to make both sides of Eq. (11) have the same dimension. The corresponding $SO(2)$ gauge covariant derivative for a vector field on Σ, $n^a(x)$, with respect to ω_μ^{ab} is denoted by

$$D_\mu n^a = \partial_\mu n^a - \omega_\mu^{ab} n^b, \qquad a, b = 1, 2 \tag{12}$$

In the decomposition of $SO(2)$ gauge potential [23], when $n^a(x)$ is a unit vector field

$$n^a(x) n^a(x) = 1$$

$\omega_\mu^{ab}(x)$ can be expressed in terms of $n^a(x)$ as

$$\omega_\mu^{ab} = (n^b \partial_\mu n^a - n^a \partial_\mu n^b) + (n^a D_\mu n^b - n^b D_\mu n^a)$$

which gives the $U(1)$ gauge potential decomposition

$$A_\mu = \frac{L}{2\pi} \varepsilon_{ab} (n^a \partial_\mu n^b - n^a D_\mu n^b) = \frac{L}{2\pi} (k^b \partial_\mu n^b - k^b D_\mu n^b) \tag{13}$$

where

$$k^b = \varepsilon_{ab} n^a \tag{14}$$

satisfying

$$k^a k^a = 1, \qquad k^a n^a = 0$$

i.e., k^a is also a unit vector normal to n^a. Then, for any given 2-dimensional unit vector $v^a(x)$ on Σ with

$$v^a n^a = \cos\theta, \qquad v^a k^a = \sin\theta, \qquad v^a v^a = 1$$

it can be expanded by $n^a(x)$ and $k^a(x)$ as

$$v^a = n^a \cos\theta + k^a \sin\theta \tag{15}$$

If $v^a(x)$ is a gauge parallel unit vector, i.e.,

$$D_\mu v^a = 0$$

the $SO(2)$ and $U(1)$ gauge potentials can be rewritten in terms of $v^a(x)$ as

$$\omega_\mu^{ab} = \partial_\mu v^a v^b - \partial_\mu v^b v^a \tag{16}$$

$$A_\mu = \frac{L}{2\pi} \varepsilon_{ab} v^a \partial_\mu v^b \tag{17}$$

These two decompositions were used to discuss the topological problems in our previous work (e.g., refs. 14, 16, 17, 24). Multiplying (16) by $k^a n^b$ and using the expansion (15), we can derive an important formula

$$k^a(\partial_\mu n^a - \omega_\mu^{ab} n^b) = -\partial_\mu \theta$$

that is,

$$k^a D_\mu n^a = -\partial_\mu \theta$$

Substituting this formula and (14) into (13), we can further express $A_\mu(x)$ in the form

$$A_\mu = \frac{L}{2\pi} \varepsilon_{ab} n^a \partial_\mu n^b + \frac{L}{2\pi} \partial_\mu \theta \tag{18}$$

in which the term $(L/2\pi)\partial_\mu\theta$ can be looked upon as the $U(1)$-like gauge transformation by comparing this expression with (7). From (17) and (18) we see that the $U(1)$ gauge potential decomposition through a non-gauge parallel unit vector differs from that of a gauge parallel unit vector only in a $U(1)$-like gauge transformation, which has no contribution to the gauge field strength $T_{\mu\nu}$. Therefore, the $U(1)$ gauge potential decomposition can be just taken as

$$A_\mu = \frac{L}{2\pi}\varepsilon_{ab}n^a\partial_\mu n^b \tag{19}$$

Here, one must notice that, though the decomposition (19) takes the pure gauge form, it is not nonsense when $n^a(x)$ has some singularities. In fact, it is just the singularities of $n^a(x)$ that contribute to the topological invariant Q in Riemann–Cartan manifold. This point can be seen in the next section. From (19) the gauge field strength $T_{\mu\nu}$ and the topological invariant Q are changed into

$$T_{\mu\nu} = \frac{L}{2\pi}\varepsilon_{ab}(\partial_\mu n^a\partial_\nu n^b - \partial_\nu n^a\partial_\mu n^b) \tag{20}$$

and

$$Q = \frac{L}{2\pi}\oint_\Sigma \varepsilon_{ab}\partial_\mu n^a\partial_\nu n^b dx^\mu \wedge dx^\nu \tag{21}$$

respectively. The consequences (19)–(21) represent a great improvement respect to our previous work (see, for example, refs. 14, 16, 17, 24) because in (19)–(21) the unit vector $n^a(x)$ is no longer required to be a gauge parallel.

4. THE INNER STRUCTURE OF THE TOPOLOGICAL INVARIANT

In this section, we will investigate the inner struction of Q through its topological quantization. Since $n^a(x)$ is a unit vector field, it can, in general, be further expressed as follows [24]:

$$n^a(x) = \frac{\phi^a(x)}{\|\phi(x)\|}, \qquad \|\phi(x)\| = \sqrt{\phi^a(x)\phi^a(x)}$$

where $\phi^a(x)$ $(a = 1, 2)$ is a vector field on Σ, i.e.,

$$\phi^a(x) = \phi^a(x^\mu(u^1, u^2)) = \phi^a(u^1, u^2)$$

In dislocation continuum theory, $n^a(x)$ stands for the order parameter describing the defects and its singularities are determined by dislocations. Obviously, the zeros of $\phi^a(x)$ are just the singularities of $n^a(x)$. Using the so-called ϕ-

mapping method, we can rewrite the topological invariant Q in (21) in the compact δ-function form [14]

$$Q = L \oint_\Sigma D\left(\frac{\phi}{u}\right)\delta^2(\vec{\phi})du^1\, du^2 \tag{22}$$

in terms of the intrinsic coördinates u^1 and u^2 of Σ, where $D(\phi/u)$ is the usual Jacobian determinant of $\vec{\phi}$ with respect to $u = (u^1, u^2)$. It is obvious that Q does not vanish only when $\vec{\phi} = 0$, i.e., the inner structure of Q is characterized by the zeros of $\vec{\phi}$ or the singularities of n. Suppose that the vector field $\vec{\phi}$ possesses N zeros, according to the deduction of ref. 25 and the implicit function theorem [26], the isolated solutions of $\vec{\phi}(u^1, u^2) = 0$ can be expressed in terms of u^1 and u^2 as

$$u^1 = a_l^1, \qquad u^2 = a_l^2, \qquad l = 1, \ldots, N$$

when the Jacobian determinant $D(\phi/u) \neq 0$, where the subscript l represents the lth solution. Then, by means of another topological invariant, the winding number of $\vec{\phi}$ at a_l [27–30], the δ-function $\delta^2(\vec{\phi})$ can be expanded by [14]

$$\delta^2(\vec{\phi}) = \sum_{l=1}^{N} \frac{\beta_l}{\left|D(\phi/u)_{a_l}\right|}\delta(u^1 - a_l^1)\delta(u^2 - a_l^2)$$

where the positive integer β_l is the absolute value of winding number and is called the Hopf index [31] of map $u \to \phi$. Making use of this expansion of $\delta^2(\vec{\phi})$, which has the topological information β_l and is regarded as a generalization to the ordinary theory of δ-function, we finally obtain the topological invariant Q in (22) at the topological quantum level as

$$Q = \sum_{l=1}^{N} \beta_l \eta_l L \tag{23}$$

where $\eta_l = \pm 1$ is called the Brouwer degree [32] of map $u \to \phi$.

From (23) we see that Q is quantized in units of a constant L, which has the dimension of length, and, going a little further, can be viewed as the elementary length in the Riemann–Cartan manifold. The topological quantum numbers are determined by the Hopf indices and Brouwer degrees of the ϕ-mapping, i.e., the winding numbers of n (or $\vec{\phi}$) at its singularities (or zeros), all of which are topological invariants and further characterize the inner structure of Q. In space-time defects, $n^a(x)$ is the order parameter of the theory, and its singularities, i.e., the zeros of $\phi^a(x)$, are labeled by the space-time dislocations. From (10) and (23), this topological invariant connects the topology of the space-time defects to their geometry. In particular, Q is constructed by means of the torsion tensor and then, in a torsion-free theory (i.e., the usual Riemannian manifold V_4), it does not exist due to the vanishing of torsion.

5. THE TOPOLOGICAL QUANTIZATION OF SPACE-TIME DEFECTS

In the early universe or the Planck era, we suggest to use the $U(1)$ gauge field strength $T_{\mu\nu}$ and the topological invariant Q to measure the size of dislocations in Riemann–Cartan manifold. We start by extending the gauge parallel vector $u^A(x)$ in Section 2 to a gauge parallel basis in internal space.

Any integral curve of ordinary differential equation (5) is determined by a point $p_0(x_0^1, \ldots, x_0^4)$ and a direction at p_0 [33]. If, at the same point p_0, we give four linearly independent directions $u_{(i)}^{\lambda}(p_0) = (dx^{\lambda}/ds_i)_{p_0}$ with

$$g_{\mu\nu}u_{(i)}^{\mu}(p_0)u_{(j)}^{\nu}(p_0) = \delta_{(ij)}, \qquad i, j = 1, 2, 3, 4$$

we obtain four geodesics and four corresponding linearly independent gauge parallel vectors marked by the index (i) $(i = 1, 2, 3, 4)$,

$$D_\mu u_{(i)}^A = 0$$

where

$$u_{(i)}^A = e_\lambda^A u_{(i)}^\lambda, \qquad u_{(i)}^A u_{(j)}^A = \delta_{(ij)}$$

is called the gauge parallel basis in internal space. The projections of the torsion tensor (4) on the basis and the corresponding topological invariants are

$$T_{\mu\nu}^{(i)} = T_{\mu\nu}^A u_{(i)}^A = \partial_\mu A_\nu^{(i)} - \partial_\nu A_\mu^{(i)}, \qquad A_\mu^{(i)} = e_\mu^A u_{(i)}^A$$

and

$$l^{(i)} = \oint_\Sigma \frac{1}{2} T_{\mu\nu}^{(i)} dx^\mu \wedge dx^\nu \tag{24}$$

respectively. In the present case, one can also show the invariant property of $l^{(i)}$ with respect to the general coordinate and local Lorentz transformations, i.e.,

$$l'^{(i)} = \oint_\Sigma \frac{1}{2} T'^{(i)}_{\mu'\nu'} dx^{\mu'} \wedge dx^{\nu'} = \oint_\Sigma \frac{1}{2} T_{\mu\nu}^{(i)} dx^\mu \wedge dx^\nu = l^{(i)}$$

by using

$$u_{(i)}^{\lambda'} = \frac{\partial x^{\lambda'}}{\partial x^\lambda} u_{(i)}^\lambda, \qquad u'^A_{(i)} = L_B^A u_{(i)}^B$$

$$T'^A_{\mu'\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\nu}{\partial x^{\nu'}} L_B^A T_{\mu\nu}^B, \qquad T'^{(i)}_{\mu'\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\nu}{\partial x^{\nu'}} T_{\mu\nu}^{(i)}$$

Since $l^{(i)}$ in (24) has the dimension of length, which leads us to call the topological invariants also the space-time dislocations in Riemann–Cartan manifold, the invariant time t is defined in analogy with (3) as

$$t = \frac{1}{c}\oint_{\Sigma} \frac{1}{2} T^{(4)}_{\mu\nu} dx^{\mu} \wedge dx^{\nu}$$

From the discussions in Sections 3 and 4 we immediately get the topological quantizations of length and time at the invariant level as

$$l^{(i)} = \sum_{l=1}^{N} \beta_l^{(i)} \eta_l^{(i)} L_{(i)} \tag{25}$$

$$t = \sum_{l=1}^{N} \beta_l^{(4)} \eta_l^{(4)} T_{(4)}, \qquad T_{(4)} = \frac{L_{(4)}}{c} \tag{26}$$

that is, the space-time dislocations are quantized in units of four elementary lengths $L_{(i)}$ ($i = 1, 2, 3, 4$). In the early universe or the Planck era, we think that the space-time is isotropic and the four elementary lengths $L_{(i)}$ can be taken as the Planck length L_p, i.e.,

$$L_{(1)} = L_{(2)} = L_{(3)} = L_{(4)} = L_p \tag{27}$$

The employment of the Planck length L_p is based on the two facts that (i) the Planck length L_p is a fundamental constant with the dimension of length and is constructed by three fundamental constants c, G, and $\hbar$, which play important roles in general relativity and quantum theory; and (ii) since torsion is linked to spin and the spin is quantized, the Planck length L_p enters through the minimal unit of spin, or action $\hbar$. On the other hand, if we change viewpoint, we see that the employment of (27) has a profound significance. Since in Riemann–Cartan manifold, due to the existence of torsion, it is shown in (25) and (26) that there must exist minimal units of length, which are taken to be the Planck length L_p, then, with the velocity of light c and the gravitational constant G, we can build up a new action-dimensional constant $\hbar = L_p^2 c^3/G$, which acts as the minimal unit of spin and leads to the construction of quantum theory. In fact, from this viewpoint, the quantization of spin can be derived directly from torsion, as will be shown in a subsequent work. By substituting the formula (27) into (25) and (26), we get

$$l^{(i)} = \sum_{l=1}^{N} \beta_l^{(i)} \eta_l^{(i)} L_p \tag{28}$$

$$t = \sum_{l=1}^{N} \beta_l^{(4)} \eta_l^{(4)} T_p, \qquad T_p = \frac{L_p}{c} \tag{29}$$

So, with torsion, we have minimum units of length and, especially, time $\neq 0$! This in fact would give us the smallest definable unit of time as $T_p \approx 10^{-43}$ sec. In the limit of $\hbar \Rightarrow 0$ (classical geometry of general relativity) or $c \Rightarrow \infty$ (Newtonian case), we recover the unphysical L_p, $T_p \Rightarrow 0$ of classical cosmology or physics.

Our suggestion (24) is invariant under general coordinate, local Lorentz, and the $U(1)$-like gauge transformations, which are not possessed in (1), and the quantizations of length and time are natural and rigorous results in our discussion. But what was dealt with in ref. 13 can only be looked upon as an assumption and the author cannot tell us how to determine the quantum numbers. On the contrary, from (28) and (29), we see that the quantum numbers are given by the Hopf indices and the Brouwer degrees, i.e., the winding numbers, which are topological invariants.

6. CONCLUSION

In Riemann–Cartan manifold, a topological invariant (constructed by means of the torsion tensor) is obtained in analogy with the theory of magnetic monopoles. It is invariant under general coordinate transformations as well as local Lorentz transformation and thus is independent of the coordinate system. Furthermore, there is another $U(1)$-like gauge invariance in it. Using the so-called ϕ-mapping method and the complete decomposition of $U(1)$ and $SO(2)$ gauge potentials, the topological invariant is formulated by a unit vector field, which need not to be gauge parallel, and is quantized naturally and rigorously in units of an elementary length. The quantum numbers are determined by the Hopf indices and Brouwer degrees. This topological invariant is closely related to the first Chern class and its inner structure is labeled by the winding numbers. In the early universe or the Planck era, in order to describe the space-time defects in an invariant form and quantize them naturally, four corresponding invariants are introduced to measure the size of space-time dislocations in Riemann–Cartan manifold by extending to a gauge parallel basis in internal space. For the above-mentioned reasons, the Planck length L_p and $T_p \approx 10^{-43}$ sec play the roles of elementary length and unit time, respectively, which has a profound significance in general relativity and quantum theory and will be detailed elsewhere.

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